TWO DESCRIPTIONS OF THE QUANTUM AFFINE ALGEBRA $U_v(\widehat{\mathfrak{sl}}_2)$ VIA HALL ALGEBRA APPROACH

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ABSTRACT. We compare the reduced Drinfeld doubles of the composition subalgebras of the category of representations of the Kronecker quiver $\overrightarrow{Q}$ and of the category of coherent sheaves on $\mathbb{P}^1$. Using this approach, we show that the Drinfeld–Beck isomorphism for the quantized enveloping algebra $U_v(\widehat{\mathfrak{sl}}_2)$ is a corollary of an equivalence between the derived categories $D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big)$ and $D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$. This technique also allows to reprove several technical results on the integral form of $U_v(\widehat{\mathfrak{sl}}_2)$.

1. INTRODUCTION

In this article, we study the relation between the composition algebra of the category of representations of the Kronecker quiver

$$\overrightarrow{Q} = \bullet \rightrightarrows \bullet$$

and the composition algebra of the category of coherent sheaves on the projective line $\mathbb{P}^1$. As it was shown by Ringel [32] and Green [17], the generic composition algebra of the category of representations of $\overrightarrow{Q}$ is isomorphic to the positive part of the quantum affine algebra $U_v(\widehat{\mathfrak{sl}}_2)$ written in the terms of Drinfeld–Jimbo generators.

On the other side, as it was discovered by Kapranov [22] and extended by Baumann and Kassel [2], the Hall algebra of the category of coherent sheaves on a projective line $\mathbb{P}^1$ is closely related with Drinfeld's new realization $U_v\big(\mathfrak{L}\mathfrak{sl}_2\big)$ of the quantized enveloping algebra of $\widehat{\mathfrak{sl}}_2$ [14]. Since then, this subject drew attention of many authors, see for example [43, 38, 25, 20, 42, 35].

In this article, we work out this important observation a step further and show that the Drinfeld–Beck isomorphism $U_v\big(\widehat{\mathfrak{sl}}_2\big) \to U_v\big(\mathfrak{L}\mathfrak{sl}_2\big)$ (see [14, 3, 27]) can be viewed as a corollary of the derived equivalence $D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big) \to D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$. The understanding of this isomorphism is of great importance for the representation theory of $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ and its applications in mathematical physics, see for example [26]. Indeed, since $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ is a Hopf algebra, the category of its finite-dimensional representations has a structure of a tensor category. However, in order to describe such representations themselves, it is frequently more convenient to work with Drinfeld's new realization $U_v\big(\mathfrak{L}\mathfrak{sl}_2\big)$.

Notation. Throughout the paper, $k = \mathbb{F}_q$ is a finite field with q elements and $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}_q = \mathbb{Q}[v, v^{-1}]/(v^{-2} - q) \cong \mathbb{Q}[\sqrt{q}]$. Next, $\mathcal{P}$ is the set of all integers of the form p^t, where p is a prime number and $t \in \mathbb{Z}_+$. For a positive integer n we set $[n] = [n]_v = \dfrac{v^n - v^{-n}}{v - v^{-1}}$ and $[n]! = [1] \dots [n]$. We denote by R the subring of the field $\mathbb{Q}(v)$ consisting of the rational functions having poles only at 0 or at roots of 1. For the affine Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ we denote by $U_v(\mathfrak{g})$ the quantized enveloping algebra over the ring R, whereas $U_q(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_R \widetilde{\mathbb{Q}}$.

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2. Hereditary categories, their Hall algebras and Drinfeld doubles

Let A be an essentially small hereditary abelian k-linear category such that for all objects $M, N \in \mathrm{Ob}(\mathsf{A})$ the k–vector spaces $\mathsf{Hom}_{\mathsf{A}}(M, N)$ and $\mathsf{Ext}^1_{\mathsf{A}}(M, N)$ are finite dimensional. In what follows, we shall call such a category *finitary*. Let $\mathsf{J} = \mathsf{J}_{\mathsf{A}} := \big(\mathrm{Ob}(\mathsf{A})/\cong\big)$ be the set of isomorphy classes of objects in A. For an object $X \in \mathrm{Ob}(\mathsf{A})$, we denote by $[X]$ its image in J. Fix the following notations.

- For any object $X \in \mathrm{Ob}(\mathsf{A})$ we set $a_X = \big|\mathsf{Aut}_{\mathsf{A}}(X)\big|$.
- For any three objects $X, Y, Z \in \mathrm{Ob}(\mathsf{A})$ we denote

$$P^Z_{X,Y} = \left|\left\{(f, g) \in \mathsf{Hom}_{\mathsf{A}}(Y, Z) \times \mathsf{Hom}_{\mathsf{A}}(Z, X) \;\middle|\; 0 \to Y \xrightarrow{f} Z \xrightarrow{g} X \to 0 \quad \text{is exact}\right\}\right|.$$

- Finally, we put $F^Z_{X,Y} = \dfrac{P^Z_{X,Y}}{a_X \cdot a_Y}$.

Note that the numbers a_Z, $P^Z_{X,Y}$, $F^Z_{X,Y}$ and $\dfrac{P^Z_{X,Y}}{a_Z}$ depend only on the isomorphy classes of X, Y, Z and are integers.

Let $K = K_0(\mathsf{A})$ be the K–group of A. For an object $X \in \mathrm{Ob}(\mathsf{A})$, we denote by $\bar{X}$ its image in K. Next, let $\langle -, - \rangle : K \times K \to \mathbb{Z}$ be the Euler form:

$$\langle \bar{X}, \bar{Y} \rangle = \mathsf{dim}_k \,\mathsf{Hom}_{\mathsf{A}}(X, Y) - \mathsf{dim}_k \,\mathsf{Ext}^1_{\mathsf{A}}(X, Y)$$

and $(-, -) : K \times K \to \mathbb{Z}$ its symmetrization:

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle, \quad \alpha, \beta \in K.$$

Following Ringel [32], one can attach to a finitary hereditary category A an associative algebra $H(\mathsf{A})$ called the extended twisted *Hall algebra* of A, defined over the field $\widetilde{\mathbb{Q}}$. As a vector space over $\widetilde{\mathbb{Q}}$, we have

$$\bar{H}(\mathsf{A}) := \bigoplus_{Z \in \mathsf{J}} \widetilde{\mathbb{Q}}[Z] \quad \text{and} \quad H(\mathsf{A}) := \bar{H}(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K].$$

For a class $\alpha \in K$ we denote by K_α the corresponding element in the group algebra $\widetilde{\mathbb{Q}}[K]$. Then we have: $K_\alpha \circ K_\beta = K_{\alpha+\beta}$. Next, for $[X], [Y] \in \mathsf{J}$ the product $\circ$ is defined to be

$$[X] \circ [Y] = \sqrt{\frac{|\mathrm{Hom}(X,Y)|}{|\operatorname{\mathsf{Ext}}^1(X,Y)|}} \sum_{[Z]\in \mathsf{J}} F^Z_{X,Y}\,[Z] = v^{-\langle \bar{X}, \bar{Y}\rangle} \sum_{[Z]\in \mathsf{J}} F^Z_{X,Y}\,[Z].$$

Finally, for any $\alpha \in K$ and $[X] \in \mathsf{J}$ we have

$$K_\alpha \circ [X] = v^{-(\alpha, \bar{X})}[X] \circ K_\alpha.$$

As it was shown in [32], the product $\circ$ is associative and the element $1 := [0] \otimes K_0$ is the unit element. In what follows, we shall use the notation $[X]K_\alpha$ for the element $[X] \otimes K_\alpha \in H(\mathsf{A})$.

Let A be a finitary *finite length* hereditary category over k. By a result of Green [17], the Hall algebra $H(\mathsf{A})$ has a natural bialgebra structure, see also [33]. The comultiplication $\Delta : H(\mathsf{A}) \to H(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} H(\mathsf{A})$ and the counit $\varepsilon : H(\mathsf{A}) \to \widetilde{\mathbb{Q}}$ are given by the following formulae:

$$\Delta\big([Z]K_\alpha\big) = \sum_{[X],[Y]\in \mathsf{J}} v^{-\langle \bar{X}, \bar{Y}\rangle} \frac{P^Z_{X,Y}}{a_Z} [X]K_{\bar{Y}+\alpha} \otimes [Y]K_\alpha \quad \text{and} \quad \varepsilon\big([Z]K_\alpha\big) = \delta_{Z,0}.$$

Moreover, as it was shown by Xiao [40], the Hall algebra $H(\mathsf{A})$ is also a Hopf algebra.

Finally, there is a pairing $(\,-\,,\,-\,) : H(\mathsf{A}) \times H(\mathsf{A}) \to \widetilde{\mathbb{Q}}$ introduced by Green [17], given by the expression

$$\Big([X]K_\alpha, [Y]K_\beta\Big) = v^{-(\alpha,\beta)} \frac{\delta_{X,Y}}{a_X}.$$

This pairing is non-degenerate on $\bar{H}(\mathsf{A})$ and symmetric. Moreover, it satisfies the following properties:

$$(a \circ b, c) = \big(a \otimes b, \Delta(c)\big) \quad \text{and} \quad (a, 1) = \varepsilon(a)$$

for any $a, b, c \in H(\mathsf{A})$. In other words, it is a bialgebra pairing.

Remark 2.1. If A is not a category of finite length (for instance, if it is the category of coherent sheaves on a projective curve) then the Green's pairing $(\,-\,,\,-\,) : H(\mathsf{A}) \times H(\mathsf{A}) \to \widetilde{\mathbb{Q}}$ is still a well-defined symmetric bilinear pairing. However, the comultiplication $\Delta([X])$ is possibly an infinite sum. Nevertheless, it is possible to introduce a completed tensor product $H(\mathsf{A})\widehat{\otimes}H(\mathsf{A})$ (which is a $\widetilde{\mathbb{Q}}$–algebra) such that $\Delta : H(\mathsf{A}) \to H(\mathsf{A})\widehat{\otimes}H(\mathsf{A})$ is an algebra homomorphism and $(\Delta\otimes\mathbb{1})\circ\Delta = (\mathbb{1}\otimes\Delta)\circ\Delta$. Moreover, for any elements $a, b, c \in H(\mathsf{A})$ the expression $\big(a \otimes b, \Delta(c)\big)$ takes a finite value and the equalities $(a \circ b,\, c) = \big(a \otimes b, \Delta(c)\big), (a, 1) = \varepsilon(a)$ are fulfilled. In such a situation we say that $H(\mathsf{A})$ is a *topological bialgebra*, see [9] for further details.

From now on, let A be an arbitrary k–linear hereditary finitary abelian category. Consider the root category $\mathsf{R}(\mathsf{A}) = D^b(\mathsf{A})/[2]$. Note that $\mathsf{R}(\mathsf{A})$ has a structure of a triangulated category such that the canonical functor $D^b(\mathsf{A}) \to \mathsf{R}(\mathsf{A})$ is exact, see [28, Section 7]. Moreover, any object of $\mathsf{R}(\mathsf{A})$ splits into a direct sum $X^+ \oplus X^-$, where $X^+ \in \mathrm{Ob}(\mathsf{A})$ and $X^- \in \mathrm{Ob}(\mathsf{A})[1]$.

Our next goal is to introduce the reduced Drinfeld double of the topological bialgebra $H(\mathsf{A})$. Roughly speaking (although, not completely correctly), it is an analogue of the Hall algebra, attached to the triangulated category $\mathsf{R}(\mathsf{A})$. To define it, consider the pair of algebras $H^{\pm}(\mathsf{A})$, where we use the notation

$$H^+(\mathsf{A}) = \bigoplus_{Z\in \mathsf{J}} \widetilde{\mathbb{Q}}[Z]^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \text{ and } H^-(\mathsf{A}) = \bigoplus_{Z\in \mathsf{J}} \widetilde{\mathbb{Q}}[Z]^- \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K].$$

and $H^{\pm}(\mathsf{A}) = H(\mathsf{A})$ as $\widetilde{\mathbb{Q}}$-algebras. Let $a = [Z]K_\gamma$ and

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} = \sum_{[X],[Y]\in \mathsf{J}} v^{-\langle \bar{X},\bar{Y}\rangle} \frac{P_{X,Y}^Z}{a_Z}[X]K_{\bar{Y}+\gamma} \otimes [Y]K_\gamma.$$

Then we denote

$$\Delta(a^{\pm}) = \sum_i a_i^{(1)\pm} \otimes a_i^{(2)\pm} = \sum_{[X],[Y]\in \mathsf{J}} v^{-\langle \bar{X},\bar{Y}\rangle} \frac{P_{X,Y}^Z}{a_Z}[X]^{\pm}K_{\pm\bar{Y}+\gamma} \otimes [Y]^{\pm}K_\gamma.$$

Definition 2.2. The Drinfeld double of $H(\mathsf{A})$ with respect to the Green's pairing $(-,-)$ is the associative algebra $\widetilde{D}H(\mathsf{A})$, which is the free product of algebras $H^+(\mathsf{A})$ and $H^-(\mathsf{A})$ subject to the following relations $D(a,b)$ for all $a, b \in H(\mathsf{A})$:

$$\sum_{i,j} a_i^{(1)-} b_j^{(2)+} \big(a_i^{(2)}, b_j^{(1)}\big) = \sum_{i,j} b_j^{(1)+} a_i^{(2)-} \big(a_i^{(1)}, b_j^{(2)}\big).$$

The following proposition is well-known, see for example [21, Section 3.2] for the case of Hopf algebras and [9] for the case of topological bialgebras.

Theorem 2.3. *The multiplication morphism* mult : $H^+(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} H^-(\mathsf{A}) \longrightarrow \widetilde{D}H(\mathsf{A})$ *is a isomorphism of $\widetilde{\mathbb{Q}}$–vector spaces. Moreover, if the category A is of finite length, then $\widetilde{D}H(\mathsf{A})$ is also a Hopf algebra such that the above morphism $H^+(\mathsf{A}) \to \widetilde{D}H(\mathsf{A})$, $a \mapsto a \otimes \mathbb{1}^-$ is an injective homomorphisms of Hopf algebras.*

The following definition is due to Xiao [40].

Definition 2.4. Let A be a k–linear finitary hereditary category. The *reduced* Drinfeld double $DH(\mathsf{A})$ is the quotient of $\widetilde{D}H(\mathsf{A})$ by the two-sided ideal

$$I = \big\langle K_\alpha^+ \otimes K_{-\alpha}^- - \mathbb{1}^+ \otimes \mathbb{1}^- \mid \alpha \in K \big\rangle.$$

Note that if A is a finite length abelian category, then I is a Hopf ideal and $DH(\mathsf{A})$ is also a Hopf algebra.

Corollary 2.5. *We have an isomorphism of $\widetilde{\mathbb{Q}}$–vector spaces*

$$\mathrm{mult} : \bar{H}^+(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \bar{H}^-(\mathsf{A}) \longrightarrow DH(\mathsf{A}).$$

Next, we shall need the following well-known theorem, a first version of which dates back to Dold [13, Satz 4.3].

Theorem 2.6. *Let A be a hereditary abelian category. Then any indecomposable object of the derived category $D^b(\mathsf{A})$ is isomorphic to $X[n]$, where X is an indecomposable object of A.*

Remark 2.7. In what follows, we shall identify an object $X \in \mathrm{Ob}(\mathsf{A})$ with its image under the canonical functor $\mathsf{A} \to D^b(\mathsf{A})$.

The following important theorem was recently proven by Cramer [12].

Theorem 2.8. *Let A and B be two k-linear finitary hereditary categories. Assume one of them is artinian and there is an equivalence of triangulated categories $D^b(\mathsf{A}) \xrightarrow{\mathbb{F}} D^b(\mathsf{B})$. Let $\mathsf{R}(\mathsf{A}) \xrightarrow{\widehat{\mathbb{F}}} \mathsf{R}(\mathsf{B})$ be the induced equivalence of the root categories. Then there is an algebra isomorphism*

$$\mathbb{F} : DH(\mathsf{A}) \longrightarrow DH(\mathsf{B})$$

uniquely determined by the following property. For any object $X \in \mathrm{Ob}(\mathsf{A})$ such that $\mathbb{F}(X) \cong \widehat{X}[-n_{\mathbb{F}}(X)]$ with $\widehat{X} \in \mathrm{Ob}(\mathsf{B})$ and $n_{\mathbb{F}}(X) \in \mathbb{Z}$ we have:

$$\mathbb{F}\big([X]^{\pm}\big) = v^{n_{\mathbb{F}}(X)\left\langle \bar{X}, \bar{X}\right\rangle} \big[\widehat{X}\big]^{\overline{n_{\mathbb{F}}(X)}} K^{n_{\mathbb{F}}(X)}_{\widehat{\mathbb{F}}(X^{\pm})},$$

where $\overline{n_{\mathbb{F}}(X)} = +$ if $n_{\mathbb{F}}(X)$ is even and $-$ if $n_{\mathbb{F}}(X)$ is odd. For $\alpha \in K$ we have: $\mathbb{F}(K_\alpha) = K_{\mathbb{F}(\alpha)}$.

3. Composition algebra of the Kronecker quiver

In this section, we study properties of the composition algebra of the Kronecker quiver

$$\overrightarrow{Q} = \begin{tikzcd} 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 2 \end{tikzcd}$$

and the reduced Drinfeld double of its Hall algebra.

Definition 3.1. Consider the pair of reflection functors $\mathbb{S}^{\pm} : \mathsf{Rep}(\overrightarrow{Q}) \to \mathsf{Rep}(\overrightarrow{Q})$ defined as follows, see [6, Section 1]. For a representation

$$X = \begin{tikzcd} V \arrow[r, shift left, "A"] \arrow[r, shift right, "B"'] & W \end{tikzcd}$$

consider the short exact sequences

$$0 \longrightarrow U' \xrightarrow{\binom{C'}{D'}} V \oplus V \xrightarrow{(A,B)} W \quad \text{and} \quad V \xrightarrow{\binom{A}{B}} W \oplus W \xrightarrow{(C'',D'')} U'' \longrightarrow 0.$$

Then we have:

$$\mathbb{S}^+(X) = \Big(U' \overset{C'}{\underset{D'}{\rightrightarrows}} V \Big) \quad \text{and} \quad \mathbb{S}^-(X) = \Big(W \overset{C''}{\underset{D''}{\rightrightarrows}} U'' \Big).$$

The action of $\mathbb{S}^\pm$ on morphisms is defined using the universal property of kernels and cokernels. Note that the functor $\mathbb{S}^+$ is left exact whereas $\mathbb{S}^-$ is right exact.

The following theorem summarizes main properties of the functors $\mathbb{S}^\pm$.

Theorem 3.2. *Let $\overrightarrow{Q}$ be the Kronecker quiver, $A = k\overrightarrow{Q}$ be its path algebra and $\mathsf{A} = \mathsf{Rep}(\overrightarrow{Q}) = A - \mathsf{mod}$. Then the following properties hold:*

(1) *The functors $\mathbb{S}^+$ and $\mathbb{S}^-$ are adjoint, i.e. for any $X, Y \in \mathrm{Ob}(\mathsf{A})$ we have:*
$$\mathsf{Hom}_{\mathsf{A}}\big(\mathbb{S}^-(X), Y\big) \cong \mathsf{Hom}_{\mathsf{A}}\big(X, \mathbb{S}^+(Y)\big).$$
(2) *The derived functors $\mathbb{R}\mathbb{S}^+$ and $\mathbb{L}\mathbb{S}^-$ are also adjoint. Moreover, they are mutually inverse auto-equivalences of the derived category $D^b(\mathsf{A})$.*
(3) *Let $X \in \mathrm{Ob}(\mathsf{A})$ be an indecomposable object. Then we have:*
$$\mathbb{R}^0\mathbb{S}^+(X) = \begin{cases} \mathbb{S}^+(X) & \text{if} \quad X \ncong S_2 \\ 0 & \text{if} \quad X \cong S_2 \end{cases} \quad \text{and} \quad \mathbb{R}^1\mathbb{S}^+(X) = \begin{cases} 0 & \text{if} \quad X \ncong S_2 \\ S_1 & \text{if} \quad X \cong S_2. \end{cases}$$
Similarly, we have:
$$\mathbb{L}^0\mathbb{S}^-(Y) = \begin{cases} \mathbb{S}^-(Y) & \text{if} \quad Y \ncong S_1 \\ 0 & \text{if} \quad Y \cong S_1 \end{cases} \quad \text{and} \quad \mathbb{L}^{-1}\mathbb{S}^-(Y) = \begin{cases} 0 & \text{if} \quad Y \ncong S_1 \\ S_2 & \text{if} \quad Y \cong S_1. \end{cases}$$
(4) *In particular, the reflection functors $\mathbb{S}^-$ and $\mathbb{S}^+$ yield mutually inverse equivalences between the categories $\mathsf{Rep}\big(\overrightarrow{Q}\big)^1$ and $\mathsf{Rep}\big(\overrightarrow{Q}\big)^2$, which are the full subcategories of $\mathsf{Rep}\big(\overrightarrow{Q}\big)$ consisting of objects without direct summands isomorphic to S_1 and S_2 respectively.*
(5) *Let $\nu = \mathbb{D}\big(\mathsf{Hom}_A(-, A)\big) : A - \mathsf{mod} \to A - \mathsf{mod}$ be the Nakayama functor and $\mathbb{S} := \mathbb{L}\nu : D^b(\mathsf{A}) \to D^b(\mathsf{A})$ be its derived functor. Then we have an isomorphism*
$$\mathsf{Hom}_{D^b(\mathsf{A})}\big(X, \mathbb{S}(Y)\big) \longrightarrow \mathbb{D}\,\mathsf{Hom}_{D^b(\mathsf{A})}(Y, X),$$
functorial in both arguments, where $\mathbb{D}$ is the duality over k. In other words, $\mathbb{S}$ is the Serre functor of the triangulated category $D^b(\mathsf{A})$ in the sense of [7].
(6) *The Serre functor $\mathbb{S}$ and the reflection functors $\mathbb{S}^\pm$ are related by an isomorphism: $\mathbb{S} \cong (\mathbb{R}\mathbb{S}^+)^2[1]$.*

Proof. The first part was essentially proven in [6, Section 1]. There the authors construct two natural transformations of functors $\imath : \mathbb{S}^-\mathbb{S}^+ \to \mathbb{1}$ and $\jmath : \mathbb{1} \to \mathbb{S}^+\mathbb{S}^-$. It can be easily shown that they define mutually inverse bijections
$$\mathsf{Hom}_{\mathsf{A}}\big(\mathbb{S}^-(X), Y\big) \longleftrightarrow \mathsf{Hom}_{\mathsf{A}}\big(X, \mathbb{S}^+(Y)\big).$$

See also [1, Section VII.5] for a proof using tilting functors.

The fact that the derived functors $\mathbb{R}\mathbb{S}^+$ and $\mathbb{L}\mathbb{S}^-$ are adjoint, is a general property of an adjoint pair, see for example [23, Lemma 15.6]. For the proof that $\mathbb{R}\mathbb{S}^+$ and $\mathbb{L}\mathbb{S}^-$ are equivalences of categories, see for example [1, Section VII.5].

By Theorem 2.6, the complexes $\mathbb{R}\mathbb{S}^+(X)$ and $\mathbb{L}\mathbb{S}^-(X)$ have exactly one non-vanishing cohomology for an indecomposable object $X \in \mathrm{Ob}(\mathsf{A})$. This proves the formulae listed in the third item. The fourth statement is proven in [6, Section 1] and for the fifth we refer to [18, Section 4.6].

For a proof of the last statement, first note that the Auslander-Reiten functor

$$\tau = \mathbb{D}\,\mathsf{Ext}^1_A(\,-\,, A) : \mathsf{Rep}(\overrightarrow{Q}) \to \mathsf{Rep}(\overrightarrow{Q})$$

is isomorphic to the Coxeter functor $\mathbb{A}^+ := (\mathbb{S}^+)^2$, see [15, Section 5.3] and [39, Proposition II.3.2]. Next, the canonical transformation of functors $\mathbb{R}\mathbb{A}^+ \to (\mathbb{R}\mathbb{S}^+)^2$ is an isomorphism on the indecomposable injective modules $I(1)$ and $I(2)$, hence it is an isomorphism. Finally, by [19, Proposition I.7.4] we know that the derived functors $\mathbb{R}\tau[1]$ and $\mathbb{L}\nu$ are isomorphic.

$\square$

Remark 3.3. Let $i \in Q_0 = \{1, 2\}$ be a vertex and $P(1) = Ae_i$ be the indecomposable projective module, which is the projective cover of the simple module S_i. Then $\mathbb{L}\nu\big(P(i)\big) = \nu\big(P(i)\big) = I(i)$, where $I(i)$ is the injective envelope of S_i.

Let X be an indecomposable object of A and $B = \mathsf{End}_{\mathsf{A}}(X)$. By Serre duality, we have an isomorphism of B-bimodules

$$\mathbb{D}\big(\mathsf{Hom}_{\mathsf{A}}(X, X)\big) \longrightarrow \mathsf{Hom}_{D^b(\mathsf{A})}(X, \mathbb{S}(X)).$$

Let $w \in \mathsf{Hom}_{D^b(\mathsf{A})}\big(X, \mathbb{S}(X)\big)$ be a generator of the socle with respect to the right (or, equivalently, left) B-module structure. Then w corresponds to the canonical map $\mathsf{Hom}_{\mathsf{A}}(X, X) \longrightarrow \mathsf{Hom}_{\mathsf{A}}(X, X)/\mathsf{rad}(\mathsf{Hom}_{\mathsf{A}}(X, X)) = k$

Lemma 3.4. *Consider a distinguished triangle*

$$\mathbb{S}[-1](X) \xrightarrow{u} Y \xrightarrow{v} X \xrightarrow{w} \mathbb{S}(X)$$

given by the morphism w. Then this triangle is almost split. Moreover, if X is non-projective, then $H^i\big(\mathbb{S}(X)\big) = 0$ for $i \neq 1$ and the short exact sequence

$$0 \longrightarrow \tau(X) \xrightarrow{u} Y \xrightarrow{v} X \longrightarrow 0$$

is almost split, where $\tau(X) = \mathbb{D}\,\mathsf{Ext}^1_A(X, A) \cong H^1\big(\mathbb{S}(X)\big)$.

Proof. For the first part of the statement, see the proof of [30, Proposition I.2.3]. For the second, see [18, Section 4.7]. $\square$

Definition 3.5. An object $X \in \mathrm{Ob}(\mathsf{A})$ is called

(1) pre-projective if there exists a projective object P and $m > 0$ such that $X \cong \tau^{-m}(P)$,

(2) pre-injective if there exists an injective object I and $m > 0$ such that $X \cong \tau^m(I)$.

Recall the classification of the indecomposable objects of the category of representations of $\overrightarrow{Q}$ over an arbitrary field k, see for example [31, Section 3.2].

Theorem 3.6. *The indecomposable representations of the Kronecker quiver $\overrightarrow{Q}$ over an arbitrary field k are the following.*

(1) *Indecomposable pre-projective objects*

$$P_n = \; k^n \underset{B_n^{\mathrm{pro}}}{\overset{A_n^{\mathrm{pro}}}{\rightrightarrows}} k^{n+1}\;, \quad n \geq 0,$$

where

$$A_n^{\mathrm{pro}} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad B_n^{\mathrm{pro}} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

In particular, P_0 and P_1 are the indecomposable projective objects.

(2) *Indecomposable pre-injective objects*

$$I_n = \; k^{n+1} \underset{B_n^{\mathrm{inj}}}{\overset{A_n^{\mathrm{inj}}}{\rightrightarrows}} k^{n}\;, \quad n \geq 0,$$

where

$$A_n^{\mathrm{inj}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad \text{and} \quad B_n^{\mathrm{inj}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

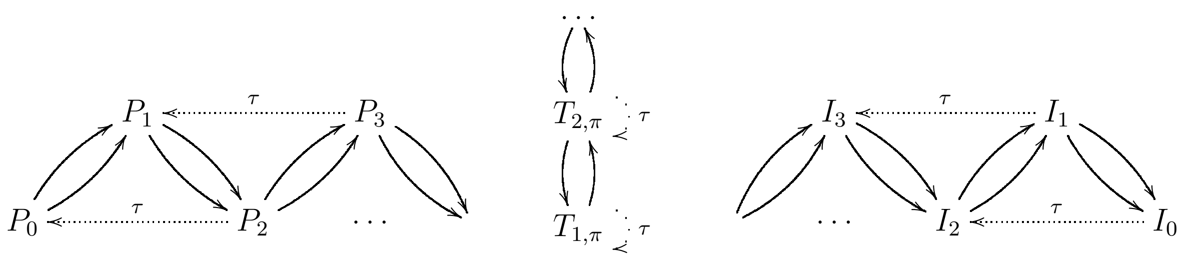
In particular, I_0 and I_1 are the indecomposable injective objects.

(3) *Tubes*

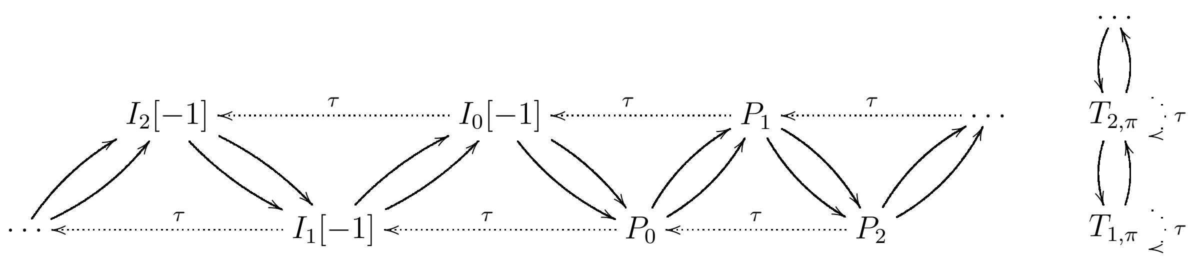
$$T_{n,\pi} = \; k^{nl} \underset{B_{n,\pi}^{\mathrm{tub}}}{\overset{A_{n,\pi}^{\mathrm{tub}}}{\rightrightarrows}} k^{nl}\;, \quad n \geq 1,$$

where $\pi(x, y) \in k[x, y]$ is an irreducible homogeneous polynomial of degree l. For $(\pi) \neq (x)$ we have: $A_{n,\pi}^{\mathrm{tub}} = I_{nl}$ and $B_{n,\pi}^{\mathrm{tub}} = F\big(\pi(1, y)^n\big)$ is the Frobenius normal form defined by the polynomial $\pi(1, y)^n$. For $(\pi) = (x)$ we set $A_{n,\pi}^{\mathrm{tub}} = F(x^n)$ and $B_{n,\pi}^{\mathrm{tub}} = I_n$.

(4) *The Auslander–Reiten quiver of* $\mathsf{Rep}(\overrightarrow{Q})$ *has the following form:*



(5) *Since* $\mathsf{Rep}(\overrightarrow{Q})$ *is hereditary, by Theorem 2.6 an indecomposable object of the derived category* $D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big)$ *is isomorphic to* $X[n]$, *where* $n \in \mathbb{Z}$ *and* X *is an indecomposable object of* $\mathsf{Rep}(\overrightarrow{Q})$. *Moreover, the Auslander–Reiten quiver of* $D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big)$ *has the shape*



Now we return to our study of Hall algebras of quiver. Applying Theorem 2.8, we get the following corollary, which is due to Sevenhant and van den Bergh [37], see also [41].

Corollary 3.7. *The derived reflection functor* $\mathbb{R}\mathbb{S}^+$ *induces an algebra isomorphism of the Drinfeld doubles:* $\mathbb{S}^+ : DH(\overrightarrow{Q}) \longrightarrow DH(\overrightarrow{Q})$, *whose inverse is the isomorphism induced by the adjoint functor* $\mathbb{L}\mathbb{S}^-$.

Definition 3.8. Let S_1 and S_2 be the simple objects of A. Consider the subalgebra $\bar{C}(\overrightarrow{Q})$ of the Hall algebra $H(\mathsf{A})$ generated by $[S_1]$ and $[S_2]$. Then the composition subalgebra is

$$C(\overrightarrow{Q}) := \bar{C}(\overrightarrow{Q}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K].$$

Note that $C(\overrightarrow{Q})$ is a Hopf subalgebra of $H(\mathsf{A})$.

Our next goal is to show the automorphisms $\mathbb{S}^{\pm}$ of $DH(\overrightarrow{Q})$ map the reduced Drinfeld double of the composition algebra $C(\overrightarrow{Q})$ to itself. In other words, one has to check that for any simple module $S_i \in \mathrm{Ob}\big(\mathsf{Rep}(\overrightarrow{Q})\big)$, $i = 1, 2$ we have: $\mathbb{S}^{\pm}\big([S_i]\big) \in DC(\overrightarrow{Q})$.

Note that in the *notations of Theorem 3.6* we have: $\mathbb{S}^+(S_1) = I_1$ is an indecomposable injective module and $\mathbb{S}^-(S_2) = P_1$ is an indecomposable projective module. Hence, it is sufficient to check the following lemma.

Lemma 3.9. *The elements $[P_1]$ and $[I_1]$ belong to the composition algebra $C(\overrightarrow{Q})$.*

Proof. Using a straightforward calculation, we get the following explicit formulae for the classes of the non-simple indecomposable projective and injective modules:

$$[P_1] = \sum_{a+b=2} (-1)^a v^{-b} [S_2]^{(a)} \circ [S_1] \circ [S_2]^{(b)} \qquad [I_1] = \sum_{a+b=2} (-1)^a v^{-b} [S_1]^{(a)} \circ [S_2] \circ [S_1]^{(b)},$$

where $[X]^{(n)} = \dfrac{[X]^n}{[n]_v!}$ for an object X of the category A. $\square$

Definition 3.10. Let $C = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ be the Cartan matrix of the affine Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. The Hopf algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ is generated over R by the elements $E_1, E_2, F_1, F_2, K_1^{\pm}, K_2^{\pm}$ subject to the relations

- the elements $Z^{\pm} := K_1^{\pm} K_2^{\pm}$ are central;
- $K_i^{\pm} K_i^{\mp} = 1 = K_i^{\mp} K_i^{\pm},\ i = 1, 2;$
- $K_i E_j = v^{-c_{ij}} E_j K_i$ and $K_i F_j = v^{c_{ij}} F_j K_i,\ i = 1, 2;$
- $[E_i, F_j] = \delta_{ij} v \dfrac{K_i - K_i^{-1}}{v - v^{-1}},\ i = 1, 2;$
- $\sum_{k=0}^{3} (-1)^k E_i^{(k)} E_j E_i^{(3-k)} = 0$ for $1 \le i \ne j \le 2$,
- $\sum_{k=0}^{3} (-1)^k F_i^{(k)} F_j F_i^{(3-k)} = 0$ for $1 \le i \ne j \le 2$.

The Hopf algebra structure is given by the following formulae:

- $\Delta(E_i) = E_i \otimes \mathbb{1} + K_i \otimes E_i$, $\Delta(F_i) = F_i \otimes K_i^{-1} + \mathbb{1} \otimes F_i$ and $\Delta(K_i) = K_i \otimes K_i$;
- $\varepsilon(E_i) = 0 = \varepsilon(F_i)$, $\varepsilon(K_i) = 1$;
- $S(E_i) = -K_i^{-1} E_i$, $S(F_i) = -F_i K_i$, $S(K_i) = K_i^{-1}$ for all $i = 1, 2$.

The following result is a special case of a general statement due to Ringel [32] and Green [17].

Theorem 3.11. *The $\widetilde{\mathbb{Q}}$–linear morphism $U_q(\mathfrak{g}) := U_v(\mathfrak{g}) \otimes_R \widetilde{\mathbb{Q}} \xrightarrow{\mathsf{ev}_q} DC(\overrightarrow{Q})$ mapping E_i to $[S_i]^+$, F_i to $[S_i]^-$ and K_i to $K_{\bar{S}_i}$ for $i = 1, 2$, is an isomorphism of Hopf algebras. Moreover, if*

$$DC_{\mathrm{gen}}(\overrightarrow{Q}) := \prod_{q \in \mathcal{P}} DC\big(\mathsf{Rep}(\mathbb{F}_q \overrightarrow{Q})\big)$$

then the R–linear map $\mathsf{ev} = \prod \mathsf{ev}_q : U_v(\mathfrak{g}) \longrightarrow DC_{\mathrm{gen}}(\overrightarrow{Q})$ is injective. The same implies to the subalgebra $U_v(\mathfrak{g}^+ \oplus \mathfrak{h})$ and the algebra $C_{\mathrm{gen}}(\overrightarrow{Q}) := \prod_{q \in \mathcal{P}} C\big(\mathsf{Rep}(\mathbb{F}_q \overrightarrow{Q})\big)$.

Remark 3.12. Note that the relations of the quantum affine algebra $U_v(\widehat{\mathfrak{sl}}_2)$ given in Definition 3.10, slightly differ from the classical ones as defined for instance in [11, 24, 26]. Namely, we impose the commutation relation $[E_i, F_i] = v \dfrac{K_i - K_i^{-1}}{v - v^{-1}}$, whereas the conventional form would be $[E_i, F_i] = \dfrac{K_i - K_i^{-1}}{v - v^{-1}}$, $i = 1, 2$. However,

one can easily pass to the conventional form replacing at the *first step* v by v^{-1} and then applying the Hopf algebra automorphism sending E_i to E_i, K_i to K_i and F_i to $-v^{-1}F_i$ for $i = 1, 2$ at the *second step*.

In order to get the relations of the Drinfeld double, which are closer to the conventional ones, one can alternatively redefine Green's form by setting

$$\Bigl([X]K_\alpha, [Y]K_\beta\Bigr)_{\text{new}} = v^{-(\alpha,\beta)+\mathsf{dim}(X)}\frac{\delta_{X,Y}}{a_X},$$

where $\mathsf{dim}(X)$ is the dimension of X over k viewed as an $k\overrightarrow{Q}$–module, and take the reduced Drinfeld double with respect to $(\,-\,,\,-\,)_{\text{new}}$.

Nevertheless, we prefer to follow the form of the relations as stated in Definition 3.10, because they seem to be more natural from the point of view of Hall algebras.

We get the following corollary, which is a special case of a result obtained by Sevenhant and van den Bergh [37], see also [41].

Corollary 3.13. *The derived functors $\mathbb{RS}^+$ and $\mathbb{LS}^-$ induce a pair of mutually inverse automorphisms $\mathbb{S}^\pm$ of the algebra $U_v(\mathfrak{g})$ such that the following diagrams are commutative:*

$$\begin{tikzcd} U_v(\mathfrak{g}) \arrow[r, "\mathbb{S}^+"] \arrow[d, "\mathsf{ev}_q"'] & U_v(\mathfrak{g}) \arrow[d, "\mathsf{ev}_q"] \\ DC(\overrightarrow{Q}) \arrow[r, "\mathbb{S}^+"] & DC(\overrightarrow{Q}) \end{tikzcd} \qquad \begin{tikzcd} U_v(\mathfrak{g}) \arrow[r, "\mathbb{S}^-"] \arrow[d, "\mathsf{ev}_q"'] & U_v(\mathfrak{g}) \arrow[d, "\mathsf{ev}_q"] \\ DC(\overrightarrow{Q}) \arrow[r, "\mathbb{S}^-"] & DC(\overrightarrow{Q}). \end{tikzcd}$$

Their action on the generators is given given by the following formulae:

$E_1 \xrightarrow{\mathbb{S}^+} \sum\limits_{a+b=2} (-1)^a v^{-b} E_1^{(a)} E_2 E_1^{(b)}$	$E_2 \xrightarrow{\mathbb{S}^-} \sum\limits_{a+b=2} (-1)^a v^{-b} E_2^{(a)} E_1 E_2^{(b)}$
$F_1 \xrightarrow{\mathbb{S}^+} \sum\limits_{a+b=2} (-1)^a v^{-b} F_1^{(a)} F_2 F_1^{(b)}$	$F_2 \xrightarrow{\mathbb{S}^-} \sum\limits_{a+b=2} (-1)^a v^{-b} F_2^{(a)} F_1 F_2^{(b)}$
$E_2 \xrightarrow{\mathbb{S}^+} v^{-1}K_1^{-1}F_1, \quad F_2 \xrightarrow{\mathbb{S}^+} vE_1K_1$	$E_1 \xrightarrow{\mathbb{S}^-} v^{-1}F_2K_2, \quad F_1 \xrightarrow{\mathbb{S}^-} vK_2^{-1}E_2$
$K_1 \xrightarrow{\mathbb{S}^+} K_1^2K_2, \quad K_2 \xrightarrow{\mathbb{S}^+} K_1^{-1}$	$K_1 \xrightarrow{\mathbb{S}^-} K_2^{-1}, \quad K_2 \xrightarrow{\mathbb{S}^-} K_1K_2^2$

As it was explained in [37, Theorem 13.1]*, in the conventions of Remark 3.12, these automorphisms coincide with the symmetries discovered by Lusztig* [24].

The following automorphism plays an important role in our study of the quantized enveloping algebra $U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr)$.

Definition 3.14. The automorphism $\mathbb{A} = \bigl(\mathbb{S}^+\bigr)^2 : DC(\overrightarrow{Q}) \longrightarrow DC(\overrightarrow{Q})$ is called Coxeter automorphism of $DC(\overrightarrow{Q})$. Using Corollary 3.13, we also obtain the corresponding automorphism of the algebra $U_v(\mathfrak{g})$, given by the commutative diagram

$$\begin{tikzcd} U_v(\mathfrak{g}) \arrow[r, "\mathbb{A}^+"] \arrow[d, "\mathsf{ev}"'] & U_v(\mathfrak{g}) \arrow[d, "\mathsf{ev}"] \\ DC_{\mathrm{gen}}(\overrightarrow{Q}) \arrow[r, "(\mathbb{S}^+)^2"] & DC_{\mathrm{gen}}(\overrightarrow{Q}). \end{tikzcd}$$

The inverse automorphism $\mathbb{A}^- = \bigl(\mathbb{S}^-\bigr)^2$ is defined in a similar way.

Finally, we have the following well-known result, see [43, 38, 20].

Lemma 3.15. *The composition algebra $C(\overrightarrow{Q})$ contains all the elements $[X]$, where X is either an indecomposable pre-injective module or an indecomposable pre-projective module.*

Proof. Let X be either pre-projective or pre-injective. Since the automorphisms $\mathbb{A}^\pm$ act on the Drinfeld double $DC(\overrightarrow{Q})$, we know that $\mathbb{A}^\pm\bigl([X]\bigr)$ belong to $DC(\overrightarrow{Q})$. As in Corollary 2.5, we have a triangular decomposition

$$DC(\overrightarrow{Q}) = \bar{C}(\overrightarrow{Q})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \bar{C}(\overrightarrow{Q})^-.$$

Hence, the element $\mathbb{A}^\pm\bigl([X]\bigr)$ must belong to one of the aisles $\bar{C}(\overrightarrow{Q})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K]$ or $\bar{C}(\overrightarrow{Q})^- \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K]$. Since any indecomposable pre-projective or pre-injective object X is of the form $\mathbb{A}^{2m}(S)$, where $m \in \mathbb{Z}$ and S is a simple object in $\mathsf{Rep}(\overrightarrow{Q})$, this implies the claim. □

4. Composition algebra of the category of coherent sheaves on $\mathbb{P}^1$

In this subsection, we consider the composition subalgebra of the category of coherent sheaves on the projective line $\mathbb{P}^1$.

First note that in this case the maps $\mathrm{Pic}(\mathbb{P}^1) \xrightarrow{\deg} \mathbb{Z}$ and $K_0\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr) \xrightarrow{(\mathrm{rk},\deg)} \mathbb{Z}^2$ are isomorphisms of abelian groups. Next, we recall some well-known facts on coherent sheaves on $\mathbb{P}^1$.

Theorem 4.1. *The indecomposable objects of the category $\mathsf{Coh}(\mathbb{P}^1)$ are:*

(1) *the line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$, $n \in \mathbb{Z}$;*
(2) *torsion sheaves $\mathcal{S}_{t,x} := \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_x^t$, where $x \in \mathbb{P}^1$ is a closed point and $t \in \mathbb{Z}_{>0}$.*

Definition 4.2. Let $\mathsf{Tor}(\mathbb{P}^1)$ be the abelian category of torsion coherent sheaves on $\mathbb{P}^1$ and $H(\mathbb{P}^1)_{\mathrm{tor}} \subseteq H(\mathbb{P}^1)$ be its Hall algebra. For any integer $r \geq 1$ consider the element

$$\mathbb{1}_{(0,r)} := \sum_{\mathcal{T} \in \mathsf{Tor}(\mathbb{P}^1):\, \bar{\mathcal{T}} = (0,r)} [\mathcal{T}] \in H(\mathbb{P}^1)_{\mathrm{tor}}.$$

Next, we introduce the elements $\{T_r\}_{r\geq 1}$ which are determined by $\mathbb{1}_{(0,r)}$ using the generating series

$$1+\sum_{r=1}^{\infty}\mathbb{1}_{(0,r)}t^r=\exp\Bigl(\sum_{r=1}^{\infty}\frac{T_r}{[r]_v}t^r\Bigr).$$

Finally, the elements $\{\Theta_r\}_{r\geq 1}$ are defined by the generating series

$$1+\sum_{r=1}^{\infty}\Theta_r t^r=\exp\bigl((v^{-1}-v)\sum_{r=1}^{\infty}T_r t^r\bigr).$$

In what follows, we set $\mathbb{1}_{(0,r)}=T_0=\Theta_0=[0]=\mathbb{1}$.

Proposition 4.3. *In the notations as above we have:*

- *The three sets elements* $\langle\mathbb{1}_{(0,r)}\rangle_{r\geq 1}$, $\langle T_r\rangle_{r\geq 1}$ *and* $\langle\Theta_r\rangle_{r\geq 1}$ *introduced in Definition 4.2, generate the same subalgebra* $U(\mathbb{P}^1)_{\mathrm{tor}}$ *of the Hall algebra* $H(\mathbb{P}^1)_{\mathrm{tor}}$*;*
- *For any* $r,s\geq 1$ *we have the equalities:*

$$\Delta(T_r)=T_r\otimes\mathbb{1}+K_{(0,r)}\otimes T_r\quad\textit{and}\quad\bigl(T_r,T_s\bigr)=\delta_{r,s}\frac{[2r]}{r(v^{-1}-v)}.$$

Proof. The first part of this proposition is trivial, a proof of the second can be found for instance in [35]. □

Using this proposition, we get the following statement.

Lemma 4.4. *In the notation as above we have:* $(\Theta_r,T_r)=\dfrac{[2r]}{r}$ *for any* $r\in\mathbb{Z}_{>0}$.

Proof. For a sequence of non-negative integers $\underline{c}=(c_r)_{r\in\mathbb{Z}_{>0}}$ such that all but finitely many entries are zero, we set:

$$T_{\underline{c}}:=\prod_{r=1}^{\infty}\frac{T_r^{c_r}}{c_r!}\quad\text{and}\quad c=\sum_{r=1}^{\infty}rc_r.$$

Then we have:

$$\Delta(T_{\underline{c}})=\sum_{\underline{a}+\underline{b}=\underline{c}}T_{\underline{a}}K_{(0,b)}\otimes T_{\underline{b}}.$$

In particular, by induction we obtain:

$$(T_c,T_{\underline{c}})=\begin{cases}\dfrac{[2c]}{c(v^{-1}-v)} & \text{if}\quad \underline{c}=(0,\dots,0,\underset{c-\text{th pl}}{1},0,\dots),\\ 0 & \text{otherwise.}\end{cases}$$

Using the formula

$$\Theta_r=(v^{-1}-v)T_r\ +\ \text{ monomials of length }\ \geq 2\ \text{ in } T_s,$$

the statement follows. □

Summing up, the first set of generators $\{\mathbb{1}_{(0,r)}\}_{r\geq 0}$ of the algebra $U(\mathbb{P}^1)_{\mathrm{tor}}$ has a clear algebro–geometric meaning. The second set $\{T_r\}_{r\geq 0}$ has a good behavior with respect to the bialgebra structure: all the generators T_r are primitive and orthogonal with respect to Green's pairing. The role of the generators Θ_r is explained by the following proposition.

Proposition 4.5. *For any $n \in \mathbb{Z}$ we have:*

$$\Delta\bigl([\mathcal{O}_{\mathbb{P}^1}(n)]\bigr) = [\mathcal{O}_{\mathbb{P}^1}(n)] \otimes \mathbb{1} + \sum_{r=0}^{\infty} \Theta_r K_{(1,n-r)} \otimes [\mathcal{O}_{\mathbb{P}^1}(n-r)].$$

Proof. We refer to [36, Section 12.2] and to [9, Lemma 5.3] for a proof of this result. □

Remark 4.6. In the next section, we shall need another description of the elements Θ_r, see [35, Example 4.12]

$$\Theta_r = v^{-r} \sum_{\substack{x_1,\dots,x_m \in \mathbb{P}^1;\, x_i \neq x_j\ 1\leq i\neq j\leq m \\ t_1,\dots,t_m :\, \sum_{i=1}^m t_i \deg(x_i) = r}} \prod_{i=1}^{m} \bigl(1 - v^{2\deg(x_i)}\bigr) [\mathcal{S}_{t_i,x_i}].$$

Definition 4.7. The the composition algebra $U(\mathbb{P}^1)$ is the subalgebra of the Hall algebra $H(\mathsf{Coh}(\mathbb{P}^1))$ generated by the elements $L_n := [\mathcal{O}_{\mathbb{P}^1}(n)]$, T_r and K_α, where $n \in \mathbb{Z}$, $r \geq 1$ and $\alpha \in K_0(\mathsf{Coh}(\mathbb{P}^1)) \cong \mathbb{Z}^2$. We also use the notations: $\delta = (0,1) \in K_0(\mathbb{P}^1)$, $C = K_\delta$ and $K = K_{(1,0)}$.

A complete list of relations between the generators of the composition algebra $U(\mathbb{P}^1)$ was obtained by Kapranov [22] and Baumann–Kassel [2], see also [35, Section 4.3].

Theorem 4.8. *The elements L_n, T_r, K and C satisfy the following relations:*

1. *C is central;*
2. *$[K, T_n] = 0 = [T_n, T_m]$ for all $m, n \in \mathbb{Z}_{>0}$;*
3. *$KL_n = v^{-2} L_n K$ for all $n \in \mathbb{Z}$;*
4. *$[T_r, L_n] = \frac{[2r]}{r} L_{n+r}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;*
5. *$L_m L_{n+1} + L_n L_{m+1} = v^2\bigl(L_{n+1} L_m + L_{m+1} L_n\bigr)$ for all $m, n \in \mathbb{Z}$.*

Let $U(\mathbb{P}^1)_{\mathrm{vec}}$ be the subalgebra of $U(\mathbb{P}^1)$ generated by the elements L_n, $n \in \mathbb{Z}$. Then the $\widetilde{\mathbb{Q}}$–linear map

$$U(\mathbb{P}^1)_{\mathrm{vec}} \otimes_{\widetilde{\mathbb{Q}}} U(\mathbb{P}^1)_{\mathrm{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \xrightarrow{\mathsf{mult}} U(\mathbb{P}^1)$$

is an isomorphism. In particular, the elements

$$B_{\underline{m},\underline{l},a,b} = \prod_{n\in\mathbb{Z}} L_n^{m_n} \circ \prod_{r\in\mathbb{Z}^+} T_r^{l_r} \circ K^a C^b,$$

where $a, b \in \mathbb{Z}$, $\underline{m} = (m_n)_{n\in\mathbb{Z}}$ and $\underline{l} = (l_r)_{r\in\mathbb{Z}_{>0}}$ are the sequences of non-negative integers such that all but finitely entries are zero, form a basis of $U(\mathbb{P}^1)$.

It turns out that in order to relate the reduced Drinfeld double $DU(\mathbb{P}^1)$ with Drinfeld's new presentation of $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ [14], one has to modify the definition of the generators $T_r^{\pm}$ and $\Theta_r^{\pm}$. Similarly to Corollary 2.5, we have a triangular decomposition

$$DU(\mathbb{P}^1) = \bar{U}(\mathbb{P}^1)^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \bar{U}(\mathbb{P}^1)^-,$$

where $K = K_0(\mathbb{P}^1) \cong \mathbb{Z}^2$ is the K–group and the element $C = K_{(0,1)}$ is central, see also [9]. Consider the group $K' := \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$ and the algebra $\widetilde{DU}(\mathbb{P}^1)$ obtained from $DU(\mathbb{P}^1)$ by adding two central generators $C^{\pm\frac{1}{2}} = K_{(0,\pm\frac{1}{2})}$ such that $C^{\frac{1}{2}}C^{-\frac{1}{2}} = \mathbb{1} = C^{-\frac{1}{2}}C^{\frac{1}{2}}$ and $\big(C^{\frac{1}{2}}\big)^2 = C$. For any $r \in \mathbb{Z}_{>0}$ we set: $\widetilde{T}_r^{\pm} = T_r^{\pm} \cdot C^{\mp\frac{r}{2}}$ and $\widetilde{\Theta}_r^{\pm} = \Theta_r^{\pm} \cdot C^{\mp\frac{r}{2}}$. Then we have:

$$\Delta\big(\widetilde{T}_r^{+}\big) = \widetilde{T}_r^{+} \otimes C^{-\frac{r}{2}} + C^{\frac{r}{2}} \otimes \widetilde{T}_r^{+}, \quad r \in \mathbb{Z}_{>0}$$

and

$$\Delta(L_n^{+}) = L_n^{+} \otimes \mathbb{1} + \mathbb{1} \otimes L_n^{+} + \sum_{r=1}^{\infty} \widetilde{\Theta}_r^{+} K C^{n-\frac{r}{2}} \otimes L_{n-r}^{+}, \quad n \in \mathbb{Z}.$$

Note that by Lemma 4.4 we have: $\big(\widetilde{\Theta}_r, \widetilde{T}_r\big) = \big(\Theta_r, T_r\big) = \dfrac{[2r]}{r}$. Using the relations of the Drinfeld double and rewriting the relations of Theorem 4.8, we obtain:

(1) $\big[K, \widetilde{T}_n^{\pm}\big] = 0 = \big[\widetilde{T}_n^{\pm}, \widetilde{T}_m^{\pm}\big]$ for all $m, n \in \mathbb{Z}_{>0}$;
(2) $KL_n^{\pm} = v^{\mp 2} L_n^{\pm} K$ for all $n \in \mathbb{Z}$;
(3) $\big[\widetilde{T}_r^{\pm}, L_n^{\pm}\big] = \dfrac{[2r]}{r} L_{n+r}^{\pm} C^{\mp\frac{r}{2}}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;
(4) $\big[L_n^{\pm}, \widetilde{T}_r^{\mp}\big] = \dfrac{[2r]}{r} L_{n-r}^{\pm} C^{\mp\frac{r}{2}}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;
(5) $\big[\widetilde{T}_r^{+}, \widetilde{T}_s^{-}\big] = \delta_{r,s} \dfrac{[2r]}{r} \dfrac{C^{-r} - C^{r}}{v^{-1} - v}$, where $r, s \in \mathbb{Z}_{>0}$;
(6) Finally, we have

$$\big[L_n^{+}, L_m^{-}\big] = \begin{cases} \dfrac{v}{v - v^{-1}}\, \widetilde{\Theta}_{n-m}^{+} K C^{\frac{m+n}{2}} & \text{if} \quad n > m, \\ 0 & \text{if} \quad n = m, \\ \dfrac{v}{v^{-1} - v}\, \widetilde{\Theta}_{m-n}^{-} K^{-1} C^{-\frac{m+n}{2}} & \text{if} \quad n < m. \end{cases}$$

Definition 4.9. Consider the R–algebra $U_v(\mathfrak{Lsl}_2)$ generated by the elements $X_n^{\pm}$ ($n \in \mathbb{Z}$), H_r ($r \in \mathbb{Z} \setminus \{0\}$), $C^{\pm\frac{1}{2}}$ and $K^{\pm}$ subject to the following relations:

(1) $C^{\frac{1}{2}}$ is central;
(2) $K^{\pm}K^{\mp} = 1 = K^{\mp}K^{\pm}$, $C^{\frac{1}{2}}C^{-\frac{1}{2}} = 1 = C^{-\frac{1}{2}}C^{\frac{1}{2}}$,
(3) $[K, H_r] = 0$ for all $r \in \mathbb{Z} \setminus \{0\}$, $KX_n^{\pm} = v^{\mp 2} X_n^{\pm} K$ for all $n \in \mathbb{Z}$;

(4) We have Heisenberg-type relations

$$[H_n, H_m] = \delta_{m+n,0}\frac{[2m]}{m}\frac{C^m - C^{-m}}{v - v^{-1}}$$

for all $m, n \in \mathbb{Z} \setminus \{0\}$;

(5) We have Hecke-type relations

$$\bigl[H_r, X_n^{\pm}\bigr] = \pm\frac{[2r]}{r} X_{n+r}^{\pm} C^{\mp\frac{|r|}{2}}$$

for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z} \setminus \{0\}$;

(6) $X_m^{\pm} X_{n+1}^{\pm} + X_n^{\pm} X_{m+1}^{\pm} = v^{\pm 2}\bigl(X_{n+1}^{\pm} X_m^{\pm} + X_{m+1}^{\pm} X_n^{\pm}\bigr)$ for all $m, n \in \mathbb{Z}$;

(7) Finally, for all $m, n \in \mathbb{Z}$ we have:

$$\bigl[X_m^+, X_n^-\bigr] = \frac{v}{v - v^{-1}}\bigl(\Psi_{m+n}^+ C^{\frac{m-n}{2}} - \Psi_{m+n}^- C^{\frac{n-m}{2}}\bigr) K^{\mathsf{sign}(m+n)},$$

where $\Psi_{\pm r}^{\pm} (r \geq 1)$ are given by the generating series

$$1 + \sum_{r=1}^{\infty} \Psi_{\pm r}^{\pm} t^r = \exp\bigl(\pm(v^{-1} - v)\sum_{r=1}^{\infty} H_{\pm r} t^r\bigr)$$

and $\Psi_{\pm r}^{\pm} = 0$ for $r < 0$.

Remark 4.10. Similarly to the case of $U_v(\widehat{\mathfrak{sl}}_2)$, our presentation of $U_v(\mathfrak{Lsl}_2)$ slightly differs from the conventional one, as used in [24, 11, 26]. In order to pass to their notation, one has to replace v by v^{-1} at the first step and then replace vX_n^- by $-X_n^-$ for all $n \in \mathbb{Z}$ at the second, see also Remark 3.12.

Proposition 4.11. *Let $U_q(\mathfrak{Lsl}_2) = U_v(\mathfrak{Lsl}_2) \otimes_R \widetilde{\mathbb{Q}}_q$. Then the map*

$$\mathsf{ev}_q : U_q(\mathfrak{Lsl}_2) \longrightarrow \widetilde{DU}(\mathbb{P}^1)$$

given by the rule: $X_n^+ \mapsto L_n^+$, $X_n^- \mapsto L_{-n}^-$ for $n \in \mathbb{Z}$, $H_r \mapsto \widetilde{T}_r^+$ for $r \in \mathbb{Z}_{>0}$ and $H_r \mapsto -\widetilde{T}_{-r}^-$ for $r \in \mathbb{Z}_{<0}$, $\Psi_r^+ \mapsto \widetilde{\Theta}_r^+$ for $r \in \mathbb{Z}_{>0}$ and $\Psi_r^- \mapsto \widetilde{\Theta}_{-r}^-$ for $r \in \mathbb{Z}_{<0}$, $K \mapsto K$ and $C^{\frac{1}{2}} \mapsto C^{\frac{1}{2}}$, is an isomorphism of algebras.

Proof. From the list of relations of Theorem 4.8 it follows that the morphism ev_q is well-defined. Next, consider the elements

$$B_{\underline{m}',\underline{l}',a,b,\underline{m}'',\underline{l}''} = \prod_{n\in\mathbb{Z}}[X_n^+]^{(m_n')} \circ \prod_{r\in\mathbb{Z}_{>0}}(T_r)^{l_r'} \circ K^a C^{\frac{b}{2}} \circ \prod_{n\in\mathbb{Z}}[X_n^-]^{(m_n'')} \circ \prod_{r\in\mathbb{Z}_{<0}}(T_r)^{l_r''} \in U_v(\mathfrak{Lsl}_2),$$

where $(m_n')_{n\in\mathbb{Z}}, (m_n'')_{n\in\mathbb{Z}}, (l_r')_{r\in\mathbb{Z}_{>0}}, (l_r'')_{r\in\mathbb{Z}_{>0}}$ run through the set of all sequences of non-negative integers such that all but finitely many entries are zero, and $a, b \in \mathbb{Z}$. Using the defining relations it is not difficult to see that all the elements $B_{\underline{m}',\underline{l}',a,b,\underline{m}'',\underline{l}''}$ generate $U_v(\mathfrak{Lsl}_2)$ as R-module. Moreover, for any $q \in \mathcal{P}$ the elements $\mathsf{ev}_q\bigl(B_{\underline{m}',\underline{l}',a,b,\underline{m}'',\underline{l}''}\bigr)$ are linearly independent in $\widetilde{DU}(\mathbb{P}^1)$. Hence, $U_v(\mathfrak{Lsl}_2)$ is free as

an R–module and $B_{\underline{m}',\underline{l}',a,b,\underline{m}'',\underline{l}''}$ is its basis over R. In particular, the morphism ev_q is an isomorphism for any $q \in \mathcal{P}$. $\square$

Corollary 4.12. *Let* $\widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) := \prod\limits_{q\in\mathcal{P}} DU\big(\mathbb{P}^1(\mathbb{F}_q)\big)$ *then the* R*–linear map*

$$\mathsf{ev} = \prod_{q\in\mathcal{P}} \mathsf{ev}_q : U_v(\mathfrak{Lsl}_2) \longrightarrow \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1)$$

is injective. Moreover, the elements $B_{\underline{m}',\underline{l}',a,b,\underline{m}'',\underline{l}''}$ *form a Poincaré-Birkhoff-Witt basis of* $U_v(\mathfrak{Lsl}_2)$ *viewed as an* R*-module, see also* [4].

Remark 4.13. Consider the functor

$$\mathbb{A} = \mathcal{O}_{\mathbb{P}^1}(-2) \otimes - : \quad D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \longrightarrow D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big).$$

Then it induces an automorphism of the algebra $\widetilde{DU}(\mathbb{P}^1)$ preserving the subalgebra $DU(\mathbb{P}^1)$ and given by the formulae:

$$L_n^{\pm} \xrightarrow{\mathbb{A}} L_{n\mp 2}^{\pm}, \quad T_r^{\pm} \xrightarrow{\mathbb{A}} T_r^{\pm}, \quad \Theta_r^{\pm} \xrightarrow{\mathbb{A}} \Theta_r^{\pm}, \quad \widetilde{T}_r^{\pm} \xrightarrow{\mathbb{A}} \widetilde{T}_r^{\pm}, \quad \widetilde{\Theta}_r^{\pm} \xrightarrow{\mathbb{A}} \widetilde{\Theta}_r^{\pm}$$

and

$$C^{\frac{1}{2}} \xrightarrow{\mathbb{A}} C^{\frac{1}{2}}, \quad K \xrightarrow{\mathbb{A}} KC^{-2}.$$

In particular, it corresponds to the automorphism $\mathbb{A}$ of the algebra $U_v(\mathfrak{Lsl}_2)$ such that

$$X_n^{\pm} \xrightarrow{\mathbb{A}} X_{n\mp 2}^{\pm}, \quad H_r \xrightarrow{\mathbb{A}} H_r, \quad \Psi_r \xrightarrow{\mathbb{A}} \Psi_r, \quad C^{\frac{1}{2}} \xrightarrow{\mathbb{A}} C^{\frac{1}{2}}, \quad K \xrightarrow{\mathbb{A}} KC^{-2}.$$

and such that the following diagram is commutative:

$$\begin{tikzcd}
U_v(\mathfrak{Lsl}_2) \arrow[r, "\mathbb{A}"] \arrow[d, "\mathsf{ev}"'] & U_v(\mathfrak{Lsl}_2) \arrow[d, "\mathsf{ev}"] \\
\widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) \arrow[r, "\mathbb{A}"] & \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1).
\end{tikzcd}$$

5. Categorification of Drinfeld-Beck isomorphism for $U_v\big(\widehat{\mathfrak{sl}}_2\big)$

In this subsection, we elaborate a connection between the reduced Drinfeld doubles $DU(\overrightarrow{Q})$ and $DU(\mathbb{P}^1)$. For this, recall the following result, which is due to Beilinson [5], see also [16].

Theorem 5.1. *Let* $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ *and* $A = \mathsf{End}_{\mathbb{P}^1}(\mathcal{F})$*. Then the derived functor*

$$\mathbb{F} := \mathbb{R}\,\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{F}, -\big) : D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \longrightarrow D^b\big(\mathsf{mod} - A\big)$$

is an equivalence of triangulated categories. Identifying the category $\mathsf{mod} - A$ *with the category of representations of the Kronecker quiver* $\overrightarrow{Q}$ *we have:*

(1) $\mathbb{F}$ *induces an equivalence between the category* $\mathsf{Tor}(\mathbb{P}^1)$ *of torsion coherent sheaves on* $\mathbb{P}^1$ *and the category* $\mathsf{Tub}(\overrightarrow{Q})$ *of* $\mathsf{Rep}(\overrightarrow{Q})$*, which is the additive closure of the modules lying in tubes;*
(2) $\mathbb{F}\big(\mathcal{O}_{\mathbb{P}^1}(n)\big) \cong P_{n+1}$ *if* $n \ge 0$ *and* $I_{-n}[-1]$ *if* $n < 0$.
(3) *Moreover, in the following diagram of functors*

$$\begin{tikzcd} D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \arrow[r, "\mathbb{F}"] \arrow[d, "\mathcal{O}_{\mathbb{P}^1}(\mp 2)\otimes -"'] & D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big) \arrow[d, "(\mathbb{RS}^{\pm})^2"] \\ D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \arrow[r, "\mathbb{F}"] & D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big) \end{tikzcd}$$

both compositions are isomorphic.

Proof. The result that $\mathbb{F}$ is an equivalence of categories is due to Beilinson [5], see also [16]. Next, the functor $\mathbb{A} = \mathcal{O}_{\mathbb{P}^1}(-2) \otimes -$ is the Auslander-Reiten translate in $D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$, see [7, 30]. On the other hand, by Theorem 3.2 the functor $(\mathbb{RS}^{\pm})^2$ is the Auslander-Reiten translate in $D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big)$. Since $\mathbb{F}$ is an equivalence of categories, we have: $\mathbb{F} \circ \mathbb{A} \cong (\mathbb{RS}^{+})^2 \circ \mathbb{F}$, see for example [30, Proposition I.2.3]. Next, the triangle

$$\mathcal{O}_{\mathbb{P}^1}(n-1) \xrightarrow{(-x\,y)} \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 2} \xrightarrow{\binom{y}{x}} \mathcal{O}_{\mathbb{P}^1}(n+1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n-1)[1]$$

is almost split in $D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$ for all $n \in \mathbb{Z}$. Here we use an identification

$$\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)\big) = k[x,y]_{m-n},$$

where $k[x,y]_{m-n}$ is the vector space of the homogeneous polynomials of degree $m-n$ is variables x and y. Since $\mathbb{F}$ maps almost split triangles to almost split triangles, we just have to apply Theorem 3.6 to describe the images of lines bundles under the functor $\mathbb{F}$. $\square$

Applying Theorem 2.8 we get:

Corollary 5.2. *The assignment* $\mathbb{F} : DH(\mathbb{P}^1) \longrightarrow DH(\overrightarrow{Q})$ *is an isomorphism of algebras.*

Our next goal is to show this isomorphism restricts on the isomorphism between the reduced Drinfeld doubles of the composition subalgebras $DU(\mathbb{P}^1) \longrightarrow DC(\overrightarrow{Q})$. For this it is convenient to consider a functor $\mathbb{G} : D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big) \longrightarrow D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$, which is quasi-inverse to $\mathbb{F}$.

Theorem 5.3. *The algebra isomorphism* $\mathbb{G} : DH(\overrightarrow{Q}) \longrightarrow DH(\mathbb{P}^1)$ *restricts on the isomorphism of the reduced Drinfeld doubles of the composition subalgebras* $DC(\overrightarrow{Q}) \xrightarrow{\mathbb{G}} DU(\mathbb{P}^1)$.

Proof. We use the notation $E_i = [S_i]^+$, $F_i = [S_i]^-$ and $K_i = K_{\bar{S}_i}$, $i = 1, 2$ for the elements of the algebra $DC(\overrightarrow{Q})$. Then we have:

$$E_1 \xrightarrow{\mathbb{G}} v^{-1}L^-_{-1}KC^{-1},\ E_2 \xrightarrow{\mathbb{G}} L^+_0,\ F_1 \xrightarrow{\mathbb{G}} v^{-1}L^+_{-1}K^{-1}C,\ F_2 \xrightarrow{\mathbb{G}} L^-_0,$$

$$K_1 \xrightarrow{\mathbb{G}} K^{-1}C,\ K_2 \xrightarrow{\mathbb{G}} K.$$

This implies that the image of the subalgebra $DC(\overrightarrow{Q})$ is contained in $DU(\mathbb{P}^1)$. Moreover, from the commutativity of the diagram

$$\begin{tikzcd} DC(\overrightarrow{Q}) \arrow[r, "\mathbb{G}"] \arrow[d, hook] & DU(\mathbb{P}^1) \arrow[d, hook] \\ DH(\overrightarrow{Q}) \arrow[r, "\mathbb{G}"] & DH(\mathbb{P}^1) \end{tikzcd}$$

we see that the algebra homomorphism $\mathbb{G} : DC(\overrightarrow{Q}) \longrightarrow DU(\mathbb{P}^1)$ is injective. To show it is also surjective, we have to prove that all the elements $T^{\pm}_r$ and $\Theta^{\pm}_r$ are in the image of $\mathbb{G}$ for all $r \geq 1$. Note that for any pair of integers $n > m$ we have the following relation in the reduced Drinfeld double $DU(\mathbb{P}^1)$:

$$\left[L^+_n, L^-_m\right] = \frac{v}{v - v^{-1}}\Theta^+_{n-m}KC^m.$$

This implies that $\Theta^{\pm}_r$ belong to $\mathbb{G}\big(DC(\overrightarrow{Q})\big)$ for all $r > 0$. Hence, the elements $T^{\pm}_r$ belong to $\mathbb{G}\big(DC(\overrightarrow{Q})\big)$ for all $r > 0$. This shows the surjectivity of the algebra homomorphism $\mathbb{G} : DC(\overrightarrow{Q}) \longrightarrow DU(\mathbb{P}^1)$. □

As an application of the developed technique, we get a short proof of the following formula, obtained for the first time by Szántó [38, Theorem 4.3], see also [20, Theorem 13].

Theorem 5.4. *In the Hall algebra of the Kronecker quiver $\overrightarrow{Q}$ the following identity holds for any $m, n \in \mathbb{Z}_{\geq 0}$:*

$$[I_m]\cdot[P_n]-v^2[P_n]\cdot[I_m] = \frac{v^{-(m+n+1)}}{v^{-1} - v} \sum_{\substack{\pi_1,\dots,\pi_l\in\mathcal{Q};\, \pi_i\neq\pi_j\ 1\leq i\neq j\leq l \\ t_1,\dots,t_l\,:\,\sum_{i=1}^{l} t_i \deg(\pi_i)=m+n+1}} \prod_{i=1}^{l}\big(1-v^{2\deg(\pi_i)}\big)\big[\mathcal{T}_{t_i,\pi_i}\big],$$

where we sum over the set $\mathcal{Q}$ of all homogeneous prime ideals of height one in the ring $k[x,y]$. In particular the right-hand side of this formula depends only on the sum $m + n$.

Proof. Let m and n be non-negative integers such that $m + n + 1 = r$ Then we have the following identity in the reduced Drinfeld double $DU(\mathbb{P}^1)$:

$$L^-_{-m-1}L^+_n - L^+_nL^-_{-m-1} = \frac{1}{q-1}\Theta^+_r KC^{-m-1}.$$

Note that the algebra homomorphism $\mathbb{F} : DU(\mathbb{P}^1) \longrightarrow DC(\overrightarrow{Q})$ acts as follows:

$$L^-_{-m-1} \mapsto v[I_m]^+ K_1^{-m-1} K_2^{-m}, \quad L^+_n \mapsto [P_n]^+, \quad K_{(1,-t)} = KC^{-t} \mapsto K_1^{-t} K_2^{-t+1},$$

where $m, n \in \mathbb{Z}_{\geq 0}$. It remains to observe that by Remark 4.6 we have:

$$\overline{\Theta}_r := \mathbb{F}(\Theta_r) = v^{-r} \sum_{\substack{t_1,\dots,t_l \,:\, \sum_{i=1}^l t_i \deg(\pi_i) = r \\ \pi_1,\dots,\pi_l \in \mathcal{Q};\, \pi_i \neq \pi_j \, 1 \leq i \neq j \leq l}} \prod_{i=1}^{l} \bigl(1 - v^{2\deg(\pi_i)}\bigr) \bigl[\mathcal{T}_{t_i,\pi_i}\bigr].$$

In particular, we get the equality:

$$v[I_m] K_1^{-m-1} K_2^{-m} [P_n] - v[P_n][I_m] K_1^{-m-1} K_2^{-m} = \frac{1}{q-1} \overline{\Theta}_r K_1^{-m-1} K_2^{-m}.$$

Taking into account the fact that $K_1 K_2$ is central and $K_1^{-1}[P_n] = v^{-2}[P_n] K_1^{-1}$, we end up precisely with Szántó's formula. $\square$

Another application of our approach is the following important result, which was stated by Drinfeld [14] and proven by Beck [3], see also [27].

Theorem 5.5. *We have an injective homomorphism of R–algebras*

$$\mathbb{G} : U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr) \longrightarrow U_v(\mathfrak{Lsl}_2)$$

given by the following formulae:

$$E_1 \xrightarrow{\mathbb{G}} v^{-1} X_1^- K C^{-1},\ E_2 \xrightarrow{\mathbb{G}} X_0^+,\ F_1 \xrightarrow{\mathbb{G}} v^{-1} X_{-1}^+ K^{-1} C,\ F_2 \xrightarrow{\mathbb{G}} X_0^-,$$

and

$$K_1 \xrightarrow{\mathbb{G}} K^{-1} C,\ K_2 \xrightarrow{\mathbb{G}} K.$$

Its image is the subalgebra of $U_v(\mathfrak{Lsl}_2)$ generated by the elements $X_n^\pm, T_r, C^\pm$ and $K^\pm$. Moreover, the following diagram is commutative for any $m \in \mathbb{Z}$:

$$\begin{tikzcd} U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr) \arrow[r, "\mathbb{A}^m"] \arrow[d, "\mathbb{G}"'] & U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr) \arrow[d, "\mathbb{G}"] \\ U_v(\mathfrak{Lsl}_2) \arrow[r, "\mathbb{A}^m"] & U_v(\mathfrak{Lsl}_2). \end{tikzcd}$$

Proof. This result follows directly from the commutativity of the following diagram:

$$
\begin{tikzcd}
U_v\big(\widehat{\mathfrak{sl}}_2\big) \arrow[rrr, "\mathbb{A}^\pm"] \arrow[ddd, "\mathbb{G}"'] \arrow[dr, "\mathrm{ev}"] & & & U_v\big(\widehat{\mathfrak{sl}}_2\big) \arrow[ddd, "\mathbb{G}"] \arrow[dl, "\mathrm{ev}"'] \\
& DC_{\mathrm{gen}}(\overrightarrow{Q}) \arrow[r, "\mathbb{A}^\pm"] \arrow[d, "\mathbb{G}"'] & DC_{\mathrm{gen}}(\overrightarrow{Q}) \arrow[d, "\mathbb{G}"] & \\
& \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) \arrow[r, "\mathbb{A}^\pm"] & \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) & \\
U_v(\mathfrak{Lsl}_2) \arrow[rrr, "\mathbb{A}^\pm"] \arrow[ur, "\mathrm{ev}"] & & & U_v(\mathfrak{Lsl}_2), \arrow[ul, "\mathrm{ev}"']
\end{tikzcd}
$$

which is obtained by patching together the diagrams, constructed in Corollary 3.13, Remark 4.13 and Theorem 5.3. $\square$

Remark 5.6. In order to pass to the "conventional form" of the Drinfeld-Beck isomorphism, one has to apply the automorphisms of $U_v(\widehat{\mathfrak{sl}}_2)$ and $U_v(\mathfrak{Lsl}_2)$ described in Remark 3.12 and Remark 4.10. In that terms, the "categorical isomorphism" $\mathbb{G}$ of Theorem 5.5 is equal to the composition of the conventional one (as can be found for instance in [26]) combined with the automorphism of $U_v(\widehat{\mathfrak{sl}}_2)$ given by the rule $E_1 \mapsto -E_1$, $F_1 \mapsto -F_1$ and leaving the remaining generators unchanged.

As a final application of the technique of Hall algebras to the study of the quantum affine algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$, we shall discuss several applications to the theory of its integral form.

Definition 5.7. The integral form $U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)$ of the quantum affine algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ is the $\mathbb{Q}[v,v^{-1}]$ subalgebra of $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ generated by the elements $E_i^{(n)} = \dfrac{E_i^n}{[n]!}$, $F_i^{(n)} = \dfrac{F_i^n}{[n]!}$ for all $n \in \mathbb{Z}_{\geq 0}, i = 1,2$ and K_1 and K_2.

The following result is well-known, see for example [24].

Theorem 5.8. *The algebra $U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)$ is a Hopf algebra over $\mathbb{Q}[v,v^{-1}]$ and we have:*

$$U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big) \otimes_{\mathbb{Q}[v,v^{-1}]} R = U_v\big(\widehat{\mathfrak{sl}}_2\big).$$

Let $U_v^{\mathrm{int}}\big(\mathfrak{Lsl}_2\big) := \mathbb{G}\big(U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)\big)$. As an application of our approach, we show that certain elements of $U_v\big(\mathfrak{Lsl}_2\big)$ actually belong to $U_v^{\mathrm{int}}\big(\mathfrak{Lsl}_2\big)$. First note the following well-known fact, see for example [10].

Lemma 5.9. *The elements $X_n^{\pm(m)} \in U_v\big(\mathfrak{Lsl}_2\big)$ belong to the integral form $U_v^{\mathrm{int}}\big(\mathfrak{Lsl}_2\big)$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_{>0}$.*

Proof. Let $\mathsf{A} = \mathsf{Rep}(\overrightarrow{Q})$ and $X \in \mathrm{Ob}(\mathsf{A})$ be an object such that $\mathsf{End}_{\mathsf{A}}(X) = k$ and $\mathsf{Ext}^1_{\mathsf{A}}(X, X) = 0$. Then for any $n \in \mathbb{Z}_{\geq 0}$ we have the following equality in the Hall algebra $H(\mathsf{A})$:

$$[X^{\oplus n}] = v^{n(n-1)} \frac{[X]^n}{[n]!} = v^{n(n-1)}[X]^{(n)}.$$

Using the explicit formulae of Theorem 3.2 it is easy to see that the automorphisms $\mathbb{S}^{\pm}$ of the algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ preserve the algebra $U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)$. Moreover, we also obtain that for any indecomposable pre-projective or pre-injective object $X \in \mathrm{Ob}(\mathsf{A})$ the element $[X^{\oplus n}] = v^{n(n-1)}[X]^{(n)}$ belongs to $C_{\mathrm{gen}}(\overrightarrow{Q})$ and lyes in the image of the homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to C_{\mathrm{gen}}(\overrightarrow{Q})$. Since for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ the vector bundle $\mathcal{O}_{\mathbb{P}^1}(n)^{\oplus m}$ is isomorphic to $\mathbb{G}(X^m)$ for an appropriate shift of a pre-projective or pre-injective module X, Theorem 5.5 yields the claim. $\square$

Lemma 5.10. *For any pair of non-negative integers (a, b) the element*

$$\mathbb{1}_{(a,b)} = \sum_{[X] \in \mathsf{J} : \bar{X} = (a,b)} [X]$$

of the Hall algebra $H(\overrightarrow{Q})$ belongs to the composition subalgebra $C(\overrightarrow{Q})$.

Proof. This statement follows from the equality

$$\mathbb{1}_{(a,b)} = v^{ab\langle \bar{S}_2, \bar{S}_1\rangle}[S_2^b] \circ [S_1^a] = v^{a(a-1)+b(b-1)}[S_2]^{(b)} \circ [S_1]^{(a)}.$$

$\square$

Corollary 5.11. *For any pair of non-negative integers (a, b) we have a well-defined element $\mathbb{1}_{(a,b)} \in C_{\mathrm{gen}}(\overrightarrow{Q})$ belonging to the image of $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to C_{\mathrm{gen}}(\overrightarrow{Q})$.*

Lemma 5.12. *Let $\mathsf{A} = \mathsf{Rep}(\overrightarrow{Q})$, $\mathbb{H} = \{a \in \mathbb{C} \,|\, \mathrm{Im}(a) > 0\}$ and $Z : K_0\big(\mathsf{A}\big)\big) \longrightarrow \mathbb{C}$ be any additive group homomorphism such that for any non-zero object X of A we have: $Z(\bar{X}) \in \mathbb{H}$. For any $\alpha \in K_0(\mathsf{A})$ denote*

$$\mathbb{1}^{\mathrm{ss}}_{\alpha} = \mathbb{1}^{\mathrm{ss}}_{\alpha, Z} := \sum_{[X] \in \mathsf{J} :\, X \in \mathsf{A}^{\mathrm{ss}}_{\alpha}} [X],$$

where $\mathsf{A}^{\mathrm{ss}}_{\alpha}$ is the category of semi-stable objects of class α with respect to the stability condition Z, see [34] for the definition of a Z-semi-stable object in an abelian category A. Then we have: $\mathbb{1}^{\mathrm{ss}}_{\alpha} \in C(\overrightarrow{Q})$.

Proof. First note that the existence and uniqueness of the Harder-Narasimhan filtration [34] of an object of our abelian category A implies the following identity for an arbitrary class $\alpha \in K_0(\mathsf{A})$ and a given stability condition Z:

$$\mathbb{1}_{\alpha} = \mathbb{1}^{\mathrm{ss}}_{\alpha} + \sum_{t \geq 2} \sum_{\substack{\alpha_1 + \cdots + \alpha_t = \alpha \\ \mu(\alpha_1) \geq \cdots \geq \mu(\alpha_t)}} v^{\sum_{i<j}\langle \alpha_i \alpha_j\rangle} \mathbb{1}^{\mathrm{ss}}_{\alpha_1} \circ \cdots \circ \mathbb{1}^{\mathrm{ss}}_{\alpha_t}.$$

Since the expression on the right-hand side is a finite sum, by induction we obtain that for all classes $\alpha \in K_0(\mathsf{A})$ the element $\mathbb{1}_\alpha^{\mathrm{ss}}$ belongs to the subalgebra of $H(\overrightarrow{Q})$ generated by all the elements $\{\mathbb{1}_\beta\}_{\beta \in K_0(\mathsf{A})}$. According to Lemma 5.10, this algebra coincides with the composition subalgebra $C(\overrightarrow{Q})$, what implies the claim. □

Remark 5.13. A result of Reineke [29, Theorem 5.1] provides an explicit formula expressing the elements $\mathbb{1}_\alpha^{\mathrm{ss}}$ via $\mathbb{1}_\beta$ for an arbitrary stability function Z:

$$\mathbb{1}_{\alpha,Z}^{\mathrm{ss}} = \mathbb{1}_\alpha + \sum_{t \ge 2} (-1)^{t-1} \sum_{\substack{\alpha_1 + \dots + \alpha_t = \alpha : \forall\, 1 \le s \le t-1 \\ \mu(\alpha_1) + \dots + \mu(\alpha_s) > \mu(\alpha)}} v^{\sum_{i<j} \langle \alpha_i \alpha_j \rangle} \mathbb{1}_{\alpha_1} \circ \dots \circ \mathbb{1}_{\alpha_t}.$$

In particular, it gives an element $\mathbb{1}_\alpha^{\mathrm{ss}} = \mathbb{1}_{\alpha,Z}^{\mathrm{ss}} \in C_{\mathrm{gen}}(\overrightarrow{Q})$ belonging to the image of the algebra homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \longrightarrow C_{\mathrm{gen}}(\overrightarrow{Q})$ for any class $\alpha \in K_0(\mathsf{A})$ and a given stability condition $Z : K_0(\mathsf{A}) \longrightarrow \mathbb{H}$.

Lemma 5.14. *For any $r \in \mathbb{Z}_{>0}$ the following element of the Hall algebra $H(\overrightarrow{Q})$*

$$\widetilde{\mathbb{1}}_{(r,r)} = \sum_{X \in \mathsf{Tub}(\overrightarrow{Q}) \,:\, \bar{X} = (r,r)} [X]$$

belongs to the composition algebra $C(\overrightarrow{Q})$. Moreover, it determines an element of $C_{\mathrm{gen}}(\overrightarrow{Q})$ belonging to the image of the homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to C_{\mathrm{gen}}(\overrightarrow{Q})$.

Proof. Consider the standard stability condition on the category $\mathsf{Coh}(\mathbb{P}^1)$ given by the function $Z = (\mathrm{rk}, \deg)$. Then it determines a stability condition on the derived category $D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$ in the sense of Bridgeland [8] such that any indecomposable object of $D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$ is Z-semi-stable. In particular, all objects on the category $\mathsf{Tor}(\mathbb{P}^1)$ are semi-stable of slope 0.

Now restrict this stability condition on the $\mathsf{Rep}(\overrightarrow{Q})$ embedded in $D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$ by the functor $\mathbb{G}$. Note that we have:

$$\deg\big(m\bar{S}_1 + n\bar{S}_2\big) = m \quad \text{and} \quad \mathrm{rk}\big(m\bar{S}_1 + n\bar{S}_2\big) = n - m.$$

It is easy to see that we obtain a well-defined stability condition on $\mathsf{Rep}(\overrightarrow{Q})$ in the sense of [34] such that all the indecomposable objects of $\mathsf{Rep}(\overrightarrow{Q})$ are semi-stable. In particular, the objects of the full subcategory $\mathsf{Tub}(\overrightarrow{Q})$ are precisely the Z-semi-stable objects whose classes in the K-group lye on the ray $\mathrm{rk} = 0$. Hence, we obtain: $\widetilde{\mathbb{1}}_{(r,r)} = \mathbb{1}_{(r,r)}^{\mathrm{ss}}$ for all $r \in \mathbb{Z}_{>0}$. Applying Lemma 5.12 and Remark 5.13, we get the claim. □

We conclude this section with a new proof of the following proposition, which was obtained for the first time by Chari and Pressley in [10].

Proposition 5.15. *Let $P_r, r \in \mathbb{Z}_{>0}$, be the elements of $U_v\big(\mathfrak{Lsl}_2\big)$ given by the following generating series:*

$$1 + \sum_{r=1}^{\infty} P_r C^{-\frac{r}{2}} t^r = \exp\Big(\sum_{r=1}^{\infty} \frac{\Psi_r}{[r]} t^r\Big).$$

Then P_r belong to the algebra $U_v^{\mathrm{int}}\big(\mathfrak{Lsl}_2\big)$ for all $r \in \mathbb{Z}_{>0}$.

Proof. The elements P_r have a clear meaning in the language of the Hall algebra $H(\mathbb{P}^1)$. Indeed, by Definition 4.2 we have: $\mathsf{ev}_q(P_r) = \mathbb{1}_{r\delta}$. for any $q \in \mathcal{P}$.

On the other side, Theorem 5.1 implies that $\mathbb{1}_{r\delta} = \mathbb{G}\big(\widetilde{\mathbb{1}}_{(r,r)}\big)$ for all $r \in \mathbb{Z}_{>0}$. Since by Lemma 5.14 we know that $\widetilde{\mathbb{1}}_{(r,r)}$ belongs to the image of the algebra homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to DC_{\mathrm{gen}}(\overrightarrow{Q})$, this implies that $\mathbb{1}_{r\delta}$ belongs to the image of the algebra homomorphism $\mathsf{ev} : U_v^{\mathrm{int}}(\mathfrak{Lsl}_2) \to DU_{\mathrm{gen}}(\mathbb{P}^1)$. By Theorem 5.5, the element P_r belongs to the algebra $U_v^{\mathrm{int}}(\mathfrak{Lsl}_2)$ for all $r \in \mathbb{Z}_{>0}$. □

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